4 Limit Computing (Thursday's topic)

Please evaluate the following limit.

1. $\lim_{t\to -1} \frac{(t+1)^4}{t^4}$

$= \frac{0}{1}$

$= 0$

2. $\lim_{x\to -1} \frac{t^4}{(t+1)^4}$

$= \frac{t^4}{(t+1)^4}$

3. $\lim_{x\to 0} |x|$

$= 0$

4. $\lim_{x\to 0} \frac{|x|}{x}$

DNE

$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$

5. $\lim_{x\to 1} \frac{\frac{1}{\sqrt{x^2+3}}-\frac{1}{2}}{x^3-x^2}$ *(This problem is by no means easy, but once you complete it, there's no need to be afraid of limit problems any more!)*

$= \lim_{x\to 1} \left(\frac{1}{\sqrt{x^2+3}} - \frac{1}{2}\right) \frac{1}{x^3-x^2}$

$= \lim_{x\to 1} \frac{2-\sqrt{x^2+3}}{2\sqrt{x^2+3}} \quad \frac{1}{x^2(x-1)}$

$= \lim_{x\to 1} \frac{(2-\sqrt{x^2+3})(2+\sqrt{x^2+3})}{2\sqrt{x^2+3}\;(2+\sqrt{x^2+3})} \quad \frac{1}{x^2(x-1)}$

$= \lim_{x\to 1} \frac{4-(x^2+3)}{2\sqrt{x^2+3}\;(2+\sqrt{x^2+3})} \quad \frac{1}{x^2(x-1)}$

Notice: $4-(x^2+3) = 1-x^2 = (1-x)(1+x)$

$= \lim_{x\to 1} \frac{(\overset{-1}{\cancel{1-x}})(1+x)}{2\sqrt{x^2+3}\;(2+\sqrt{x^2+3})} \quad \frac{1}{x^2\cancel{(x-1)}}$

$= \lim_{x\to 1} \frac{-(1+x)}{2\sqrt{x^2+3}\;(2+\sqrt{x^2+3})} \quad \frac{1}{x^2}$

$= \frac{-2}{2\times 2\times 4} \cdot \frac{1}{1} = -\frac{1}{8}$

6. $\lim_{x\to+\infty} \frac{x^{2016}-9x+20}{x^{2015}+2x^{2016}-221}$

$$= \lim_{x\to+\infty} \frac{x^{2016}\left(1-\frac{9}{x^{2015}}+\frac{20}{x^{2016}}\right)}{x^{2016}\left(2+\frac{1}{x}-\frac{221}{x^{2016}}\right)}$$

$$= \frac{1}{2}$$

7. $\lim_{x\to+\infty} \frac{x^{14}+1}{1-3x^7\times 5x^2}$

$$\boxed{x^a \cdot x^b = x^{a+b}} \star$$

$$= \lim_{x\to+\infty} \frac{x^{14}+1}{-15x^9+1}$$

$$= \lim_{x\to+\infty} \frac{x^{14}\left(1+\frac{1}{x^{14}}\right)}{x^9\left(-15+\frac{1}{x^9}\right)}$$

$$= \lim_{x\to+\infty} \frac{x^5\left(1+\frac{1}{x^{14}}\right)}{-15+\frac{1}{x^9}} = +\infty$$

8. $\lim_{x\to+\infty} \frac{\sqrt{9x^4+3x}}{(2x+1)^2}$

$$= \lim_{x\to+\infty} \frac{\sqrt{x^4\left(9+\frac{3}{x^3}\right)}}{\left[x\left(2+\frac{1}{x}\right)\right]^2}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{x^4}\sqrt{9+\frac{3}{x^3}}}{x^2\left(2+\frac{1}{x}\right)^2}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{9+\frac{3}{x^3}}}{\left(2+\frac{1}{x}\right)^2} = \frac{\sqrt{9}}{2^2} = \frac{3}{4}$$

9. $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{x-1}$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(1-\frac{1}{x}\right)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(1-\frac{1}{x}\right)}$$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(1-\frac{1}{x}\right)}$$

$$= -\sqrt{2}$$

$\star\star$

$$\boxed{\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}}$$

10. $\lim_{x\to+\infty} \frac{x^2+3\sqrt{x^2+1}}{\sqrt{4x^2-5}}$ *(problem ~~6-8~~ 8-10 all came from past exams)*

$$= \lim_{x\to+\infty} \frac{x^2\left(1+\frac{3\sqrt{x^2+1}}{x^2}\right)}{\sqrt{x^2\left(4-\frac{5}{x^2}\right)}}$$

$$= \lim_{x\to+\infty} \frac{x^2\left(1+\frac{3\sqrt{x^2+1}}{x^2}\right)}{\sqrt{x^2}\sqrt{4-\frac{5}{x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x\left(1+\frac{3\sqrt{x^2+1}}{x^2}\right)}{\sqrt{4-\frac{5}{x^2}}}$$

Notice $\lim_{x\to+\infty} \frac{3\sqrt{x^2+1}}{x^2} = \lim_{x\to+\infty} \frac{3\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x^2}$

$$= \lim_{x\to+\infty} \frac{3\sqrt{1+\frac{1}{x^2}}}{x} = 0$$

So ? = DNE.

11. $\lim_{x\to+\infty} \sqrt{x^2+x} - x$

$$= \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to+\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}+x}$$

$$= \lim_{x\to+\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} = \frac{1}{2}$$